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The numerical analysis on a Volterra equation with asymptotically periodic solution

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ABSTRACT

We consider order one operational quadrature methods on a certain integro-differential equation of Volterra type on $(0, \infty)$, with piecewise linear convolution kernels. The forms of discretization solution are patterned after a continuous one of Hannsgen (1979) [2]. An l^1 remainder stability and an error bound are derived.

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1. Introduction

This paper is concerned with the linear integro-differential equation

$$u'(t) + \int_0^t a(t-s)u(s)\,ds = 0, \quad t > 0, u(0) = 1, \tag{1.1}$$

$(' = d/dt)$, where u and $a(t)$ are real-valued functions. In addition, the kernel $a(t)$ has the piecewise linear form

$$a(t) = \sum_{l=1}^{\infty} \delta_l \left(1 - \frac{\min\{t, l\}}{l}\right), \tag{1.2}$$

with

$$\delta_l \geq 0, \quad 0 < a(0) = \sum_{l=1}^{\infty} \delta_l \equiv \delta < \infty, \tag{1.3}$$

and

$$\omega = \sqrt{\delta} = 2\pi j, \quad \text{for some integers } j. \tag{1.4}$$

It is easy to see that the kernel (1.2)–(1.4) is a special case in which

$$a(t) \text{ is nonnegative, nonincreasing, and convex on } (0, \infty) \text{ with } a \in L^1(0, 1) \text{ and } a(\infty) = 0. \tag{1.5}$$

Hannsgen showed in [1] that if (1.2)–(1.4) hold, then

$$u_1(t) \equiv u(t) - \frac{2}{\gamma}\cos\omega t \to 0 \quad (t \to \infty), \tag{1.6}$$

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with $\gamma = \frac{3\delta}{\delta} = 3$. Furthermore, in the same assumptions and adding an integrable condition

$$\int_0^\infty a(t)\,\mathrm{d}t < \infty. \tag{1.7}$$

Hannsgen established in [2] that

$$\int_0^\infty (|u_1(t)| + |u_1'(t)|)\,\mathrm{d}t < \infty, \tag{1.8}$$

and demonstrated that (1.8) need not hold if (1.7) fails to hold.

Our purpose in this paper is to study discretization of the problem (1.1). The methods considered will be based on the backward Euler approximation of the equation, combined with order one operational quadrature approximating the integral. The operational quadrature methods were introduced in [3] via an operational calculus for the Laplace transform. In order to describe the operational quadrature on the problem (1.1), we introduce time step denoted by k and a subscript n referring to the time level $t_n = nk$. We denote the approximation of $u(t_n)$ by u^n. The operational quadrature backward Euler scheme for approximating (1.1) is

$$\frac{u^n - u^{n-1}}{k} + \sum_{p=1}^{n} a_{n-p}(k)u^p = 0, \quad n \geq 1,\ u^0 = 1, \tag{1.9}$$

where $a_p(k)$ are the coefficients of the power series

$$\widehat{a}\left(\frac{1-z}{k}\right) = \sum_{p=0}^{\infty} a_p(k)z^p, \tag{1.10}$$

and

$$\widehat{a}(s) = \int_0^\infty a(t)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{\delta}{s} + \frac{1}{s^2}\sum_{l=1}^{\infty}\frac{\delta_l}{l}(\mathrm{e}^{-ls} - 1) \tag{1.11}$$

is the Laplace transform of the kernel function a. The $\widehat{a}(s)$ is analytic for $\mathrm{Re}\, s > 0$ and continuous for $\mathrm{Re}\, s \geq 0$.

Multiplying (1.9) by z^n and summing from 1 to ∞, we obtain for the generating function $\widetilde{u}(z) = \sum_{n=1}^{\infty} u^n z^n$, that

$$\widetilde{u}(z) = \frac{z}{k}\widehat{u}\left(\frac{1-z}{k}\right). \tag{1.12}$$

Moreover, we note that

$$\widehat{u_1}(s) = \frac{1}{s + \widehat{a}(s)} - \frac{2s}{\gamma(s^2 + \omega^2)}, \quad (\mathrm{Re}\, s > 0) \tag{1.13}$$

and $\widehat{u_1}(\bar{s}) = \overline{\widehat{u_1}(s)}$ ($\bar{s}$ = the complex conjugate of s). $\widehat{u_1}$ can be continuously extended to $\{\mathrm{Re}\, s \geq 0, s \neq \pm \mathrm{i}\omega\}$.

For a discrete analogue of (1.6), we let

$$\bar{u}(z) = \frac{z}{k}\frac{2}{\gamma}\frac{\frac{1-z}{k}}{\left(\frac{1-z}{k}\right)^2 + \omega^2} = \bar{u}^1 z + \bar{u}^2 z^2 + \cdots + \bar{u}^n z^n + \cdots, \tag{1.14}$$

and

$$u_1^n = u^n - \bar{u}^n, \quad \text{for } n \geq 1. \tag{1.15}$$

Our first purpose is to show an l^1 remainder stability estimate, which is a discrete analogue of (1.8).

Theorem 1. *Let (1.2)–(1.4) and (1.7) hold, and u^n, $\bar{u}^n$ are defined by (1.9) and (1.14), respectively. Then u_1^n satisfies*

$$k\sum_{n=1}^{\infty} |u_1^n| \leq C, \tag{1.16}$$

where and after, C stands for a positive constant, independent of k and n, possibly with different values at different places.

Next, we shall show an l^1 remainder error estimate under the assumptions of Theorem 1, but adding a moment condition on the kernel $a(t)$

$$\int_0^\infty ta(t)\,\mathrm{d}t < \infty. \tag{1.17}$$

Theorem 2. *Assume that (1.2)–(1.4), (1.7) and (1.17) hold, and let $u(t_n)$ and u^n be the solutions of (1.1) and (1.9), respectively. Define $u_1(t_n)$ as in (1.6), and u_1^n as in (1.15). Then we have that*

$$k\sum_{n=1}^{\infty}|u_1(t_n)-u_1^n| \le Ck|\log k|. \tag{1.18}$$

We try to use Theorems 1 and 2 to study the following more interesting Volterra equation (see [4–10])

$$y_t(t)+\int_0^t a(t-s)Ly(s)\mathrm{d}s=0, \quad t>0,\ y(0)=u_0, \tag{1.19}$$

where $y_t = \partial y/\partial t$ and L is a positive self-adjoint linear operator defined on a dense subspace $\mathcal{D}(L)$ of the real Hilbert space H, with a complete eigensystem $\{\lambda_m, \varphi_m\}_{m=1}^{\infty}$, u_0 belongs to H. From [4,2] we see that the solution of (1.19) is defined by the resolvent formula

$$y(t)=\sum_{j=1}^{\infty}u(t,\lambda_j)(u_0,\varphi_j)\varphi_j, \tag{1.20}$$

where $(\cdot,\cdot)$ denotes the inner product in H, and $u(t,\lambda)$ is the solution of the scalar problem

$$u'(t,\lambda)+\lambda\int_0^t a(t-s)u(s,\lambda)\mathrm{d}s=0, \quad u(0,\lambda)=1. \tag{1.21}$$

This justifies our study for equations like (1.1). Now we set

$$\lambda_j=j^2[4\pi^2/a(0)] \quad (j=1,2,3,\ldots), \tag{1.22}$$

$$u_j(t,\omega_j)=\frac{2}{\gamma}\cos\omega_j t, \qquad \omega_j=(\lambda_j a(0))^{1/2}=2\pi j, \tag{1.23}$$

$$\omega(t)=\sum_{j=1}^{\infty}u_j(t,\omega_j)(u_0,\varphi_j)\varphi_j, \tag{1.24}$$

and define

$$U^n=\sum_{j=1}^{\infty}U^n(\lambda_j)(u_0,\varphi_j)\varphi_j, \tag{1.25}$$

$$\frac{z}{k}\widehat{u}\left(\frac{1-z}{k},\lambda_j\right)=U^1(\lambda_j)z^1+\cdots+U^n(\lambda_j)z^n+\cdots$$

and

$$\Omega^n=\sum_{j=1}^{\infty}\Omega^n(\lambda_j)(u_0,\varphi_j)\varphi_j, \tag{1.26}$$

$$\frac{z}{k}\widehat{u_j}\left(\frac{1-z}{k},\omega_j\right)=\Omega^1(\lambda_j)z^1+\cdots+\Omega^n(\lambda_j)z^n+\cdots.$$

In Theorem 1, the analog of (1.16) would be

$$k\sum_{n=1}^{\infty}\|U^n-\Omega^n\| \le c(u_o), \tag{1.27}$$

where $\|\cdot\|$ denotes the norm in H and $c(u_0)$ is a positive constant only dependent on u_0.

Our attempts to prove (1.27), using the methods of [4] and the present paper, but have not succeeded. It is an open question whether this result holds even if in the conditions of Theorem 2 on the kernel $a(t)$.

The present work discuss only a local version of (1.27), and we shall prove Theorem 1 in Section 3. The proof of Theorem 2 is given in Sections 4 and 5. Finally, Section 6 contains some further remarks on the results.

2. Preliminaries

In this section we recall some results from [2], on the Laplace transform $\widehat{u_1}(s)$ and the Fourier transform

$$\widetilde{a}(\tau) = \int_0^\infty a(t)\mathrm{e}^{-\mathrm{i}\tau t}\,\mathrm{d}t.$$

Under the assumptions of Theorem 1, $\widehat{u_1}(s)$ is continuous and bounded in $\{\mathrm{Re}\, s \geq 0\}$, and $\widetilde{u_1}(\tau)$ $(-\infty < \tau < \infty)$ is absolutely continuous with

$$\int_{-\infty}^\infty |\widetilde{u_1}'(\tau)|\,\mathrm{d}\tau \leq C < \infty. \tag{2.1}$$

The Fourier transform $\widetilde{a}(\tau)$ is cubic continuously differentiable, if (1.5), (1.7) and (1.17) hold. In particular, (1.5) and (1.7) imply that $\widetilde{a}(\tau)$ is twice continuous differentiable.

From (1.7) and (1.11) we find that

$$\frac{1}{2}\sum_{l=1}^\infty l\delta_l = \int_0^\infty a(t)\,\mathrm{d}t < \infty, \tag{2.2}$$

and

$$\widehat{a}'(s) = -\frac{\delta}{s^2} - \frac{2}{s^3}\sum_{l=1}^\infty \frac{\delta_l}{l}(\mathrm{e}^{-ls} - 1) - \frac{1}{s^2}\sum_{l=1}^\infty \delta_l\,\mathrm{e}^{-ls} \tag{2.3}$$

$$\widehat{a}''(s) = \frac{2\delta}{s^3} + \frac{6}{s^4}\sum_{l=1}^\infty \frac{\delta_l}{l}(\mathrm{e}^{-ls} - 1) + \frac{4}{s^3}\sum_{l=1}^\infty \delta_l\,\mathrm{e}^{-ls} + \frac{1}{s^2}\sum_{l=1}^\infty l\delta_l\,\mathrm{e}^{-ls}, \tag{2.4}$$

and in addition (1.17), we have

$$\frac{1}{6}\sum_{l=1}^\infty l^2\delta_l = \int_0^\infty ta(t)\,\mathrm{d}t, \tag{2.5}$$

$$\widehat{a}'''(s) = -\frac{6\delta}{s^4} - \frac{24}{s^5}\sum_{l=1}^\infty \frac{\delta_l}{l}(\mathrm{e}^{-ls} - 1) - \frac{18}{s^4}\sum_{l=1}^\infty \delta_l\,\mathrm{e}^{-ls} - \frac{6}{s^3}\sum_{l=1}^\infty l\delta_l\,\mathrm{e}^{-ls} - \frac{1}{s^2}\sum_{l=1}^\infty l^2\delta_l\,\mathrm{e}^{-ls}. \tag{2.6}$$

These three functions $\widehat{a}'(s), \widehat{a}''(s), \widehat{a}'''(s)$ are analytic for $\mathrm{Re}\, s > 0$ and continuous for $\mathrm{Re}\, s \geq 0$.

A little rearranging shows that

$$s + \widehat{a}(s) = \gamma(s - \mathrm{i}\omega) - \left(\frac{1}{s} + \frac{\delta(s + \mathrm{i}\omega)}{s^2\omega^2}\right)(s - \mathrm{i}\omega)^2 + s^{-2}\sum_{l=1}^\infty \frac{\delta_l}{l}[\mathrm{e}^{-l(s-\mathrm{i}\omega)} - 1 + l(s - \mathrm{i}\omega)]. \tag{2.7}$$

By considering the cases that $|l(s - \mathrm{i}\omega)| \geq 1$ and $|l(s - \mathrm{i}\omega)| < 1$, one sees that

$$\left|\frac{\mathrm{e}^{-l(s-\mathrm{i}\omega)} - 1 + l(s - \mathrm{i}\omega)}{l(s - \mathrm{i}\omega)^2}\right| \leq 3l, \quad (1 \leq l < \infty, \mathrm{Re}\, s \geq 0, s \neq \mathrm{i}\omega).$$

Using the dominated convergence theorem we obtain that

$$\lim_{\substack{s\to\mathrm{i}\omega \\ \mathrm{Re}\, s\geq 0}} (s - \mathrm{i}\omega)^{-2}\sum_{l=1}^\infty \frac{\delta_l}{l}[\mathrm{e}^{-l(s-\mathrm{i}\omega)} - 1 + l(s - \mathrm{i}\omega)] = \int_0^\infty a(t)\,\mathrm{d}t.$$

3. Stability

We now consider the backward Euler method (1.9), with the first-order operational quadrature (1.10). The main works in this section are to show Theorem 1 mentioned in Section 1, and two related stability estimates for the solution u^n of (1.9) and $\overline{u}^n$ defined by (1.14), respectively. Throughout this section, we assume that the kernel $a(t)$ satisfies (1.2)–(1.4) and (1.7). We begin by showing a stability estimate for $\overline{u}^n$.

Lemma 3.1. *We have*

$$k\sum_{n=1}^\infty |\overline{u}^n| \leq Ck^{-2}. \tag{3.1}$$

Proof. From the Hardy's inequality [11, pp. 48] we have the estimate

$$\sum_{n=1}^{\infty}|\overline{u}^n| \le 2k\left[\int_0^{\varepsilon}+\int_{\varepsilon}^{\frac{\varepsilon}{k}}+\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}\right]|\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta, \tag{3.2}$$

where ε is taken as an appropriately small positive constant, and

$$\overline{u}_z'(z) = \frac{2}{k\gamma}\left[\frac{\frac{1}{k}\left(\frac{1-z}{k}\right)^2+\omega^2\left(\frac{1-z}{k}\right)-\frac{z}{k}\omega^2}{\left(\left(\frac{1-z}{k}\right)^2+\omega^2\right)^2}\right]. \tag{3.3}$$

Note that, for $z = x - \mathrm{i}y$, $|x| \le 1$, $y \ge 0$, $|z| = 1$,

$$\left|\frac{1-z}{k}+\mathrm{i}\omega\right| \ge \omega, \qquad \left|\frac{1-z}{k}-\mathrm{i}\omega\right| \ge Ck\omega^2. \tag{3.4}$$

In particular, when $0 \le \theta \le \varepsilon$, we have that

$$\left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}\right| \le C\varepsilon, \qquad \left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}-\mathrm{i}\omega\right| \ge \omega - c\varepsilon > 0,$$

and

$$\int_0^{\varepsilon}|\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le \frac{2\varepsilon}{\gamma k}\left[\frac{\frac{1}{k}(c\varepsilon)^2+\omega^2c\varepsilon+\frac{1}{k}\omega^2}{\omega^2(\omega-c\varepsilon)^2}\right], \tag{3.5}$$

in other words,

$$k^2\int_0^{\varepsilon}|\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le C. \tag{3.6}$$

When $\frac{\varepsilon}{k} \le \theta \le \frac{\pi}{k}$, we obtain by simple computation that

$$\left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k} \pm \mathrm{i}\omega\right| \ge \frac{1-\cos k\theta}{k} \ge \frac{1-\cos\varepsilon}{k} > 0,$$

and

$$\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}|\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le \frac{2}{\gamma k}\,\frac{\pi-\varepsilon}{k}\left[\frac{\frac{1}{k}\frac{4}{k^2}+\omega^2\frac{2}{k}+\frac{1}{k}\omega^2}{(1-\cos\varepsilon)^4/k^4}\right], \tag{3.7}$$

so that

$$k^2\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}|\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le Ck. \tag{3.8}$$

In order to obtain a similar estimate on the interval $\left[\varepsilon, \frac{\varepsilon}{k}\right]$, we partition this interval into three sets

$$\left[\varepsilon, \frac{\varepsilon}{k}\right] = [\varepsilon, \theta_1] \cup [\theta_1, \theta_2] \cup \left[\theta_2, \frac{\varepsilon}{k}\right],$$

where

$$\theta_1 = \frac{1}{2k}\arccos\frac{1}{\sqrt{1+(k\omega)^2}} - \varepsilon < \omega - \varepsilon, \quad (k \to 0),$$

$$\theta_2 = 2\frac{1}{k}\arccos\frac{1}{\sqrt{1+(k\omega)^2}} + \varepsilon > \omega + \varepsilon, \quad (k \to 0).$$

On $[\varepsilon, \theta_1]$,

$$\left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}\right| \le 2\frac{\sin k\theta}{k} \le 2\theta \le 2(\omega-\varepsilon),$$

and

$$\left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k} - \mathrm{i}\omega\right| \ge \varepsilon,$$

which, together with (3.4), yields

$$\int_{\varepsilon}^{\theta_1} |\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le \frac{2(\theta_1-\varepsilon)}{\gamma k}\left[\frac{\frac{1}{k}4(\omega-\varepsilon)^2+2\omega^2(\omega-\varepsilon)+\frac{1}{k}\omega^2}{\omega^2\varepsilon^2}\right], \tag{3.9}$$

such that

$$k^2\int_{\varepsilon}^{\theta_1} |\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le C. \tag{3.10}$$

Similarly, on $\left[\theta_2, \frac{\varepsilon}{k}\right]$, we easily get the following estimates

$$\left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}+\mathrm{i}\omega\right| \ge \frac{\sin k\theta}{k} = \tau, \qquad \left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}-\mathrm{i}\omega\right| \ge |\tau-\omega| \ge \varepsilon,$$

and

$$\left|\left(\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}\right)\Big/\left(\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}-\mathrm{i}\omega\right)\right| \le C, \qquad \cos k\theta \ge \cos\varepsilon.$$

Then, it follows that

$$\int_{\theta_2}^{\frac{\varepsilon}{k}} |\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le \frac{2}{\gamma k}\left[\frac{1}{k}c^2+\frac{\omega^2 c}{\varepsilon}+\frac{\omega^2}{k\varepsilon^2}\right]\int_{\theta_2}^{\frac{\varepsilon}{k}}\frac{\mathrm{d}\theta}{\tau^2} \le Ck^{-2}\int_{\omega+\varepsilon}^{\infty}\frac{\mathrm{d}\tau}{\tau^2}, \tag{3.11}$$

so

$$k^2\int_{\theta_2}^{\frac{\varepsilon}{k}} |\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le C. \tag{3.12}$$

Finally, we turn to the interval $[\theta_1, \theta_2]$. Observing $\left|\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}\right| \le 6\omega+2\varepsilon$, and (3.4), we have

$$\int_{\theta_1}^{\theta_2} |\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le \frac{6\omega}{\gamma k}\left[\frac{\frac{1}{k}(6\omega+2\varepsilon)^2+\omega^2(6\omega+2\varepsilon)+\frac{1}{k}\omega^2}{\omega^2c^2k^2\omega^4}\right], \tag{3.13}$$

this is

$$k^2\int_{\theta_1}^{\theta_2} |\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le Ck^{-2}. \tag{3.14}$$

Combining (3.10), (3.12) and (3.14) we have now established that

$$k^2\int_{\varepsilon}^{\frac{\varepsilon}{k}} |\overline{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \le Ck^{-2}. \tag{3.15}$$

Together with (3.2), (3.6) and (3.8), this shows our desired estimate (3.1). Hence the proof of Lemma 3.1 is completed. □

We recall from [4, Lemma 4.1] that when (1.5) holds we have

$$\frac{1}{2\sqrt{2}}\int_0^{\frac{1}{\tau}} a(t)\mathrm{d}t \le |\widehat{a}(\mathrm{i}\tau)| \le 4\int_0^{\frac{1}{\tau}} a(t)\mathrm{d}t, \quad (\tau > 0), \tag{3.16}$$

$$|\widehat{a}'(\tau)| \le 40\int_0^{\frac{1}{\tau}} ta(t)\mathrm{d}t, \quad (\tau > 0). \tag{3.17}$$

With the help of [1, Lemma 1] we may obtain

$$\frac{1}{2\sqrt{2}}\int_0^{\frac{1}{\tau}} \mathrm{e}^{-\sigma t}a(t)\mathrm{d}t \le |\widehat{a}(\sigma+\mathrm{i}\tau)| \le 4\int_0^{\frac{1}{\tau}} a(t)\mathrm{d}t, \quad (\tau > 0, \sigma \ge 0), \tag{3.18}$$

$$|\widehat{a}_s'(\sigma+\mathrm{i}\tau)| \le 40\int_0^{\frac{1}{\tau}} ta(t)\mathrm{d}t, \quad (\tau > 0, \sigma \ge 0), \tag{3.19}$$

here $\widehat{a}_s' = \frac{\mathrm{d}\widehat{a}(s)}{\mathrm{d}s}$, (see also [12, Lemma 3.2]).

We are in position to prove the following Lemma 3.2.

Lemma 3.2. *Assume that (1.2)–(1.4) and (1.7) hold, and let u^n be the solution of (1.9). Then u^n satisfies*

$$k\sum_{n=1}^{\infty}|u^n| \le Ck^{-2}. \tag{3.20}$$

Proof. Just as in the proof of Lemma 3.1, now we can pose that

$$\sum_{n=1}^{\infty}|u^n| \le 2k\left[\int_0^{\varepsilon}+\int_{\varepsilon}^{\frac{\varepsilon}{k}}+\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}\right]|\widetilde{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta. \tag{3.21}$$

Differentiating (1.12) we have that

$$\widetilde{u}_z'(z)=\frac{1}{k}\left[\frac{\frac{1}{k}+\widehat{a}\left(\frac{1-z}{k}\right)+\frac{z}{k}\widehat{a}_s'\left(\frac{1-z}{k}\right)}{\left(\frac{1-z}{k}+\widehat{a}\left(\frac{1-z}{k}\right)\right)^2}\right]. \tag{3.22}$$

Introduce the notation

$$\sigma=\sigma(k,\theta)=\frac{1-\cos k\theta}{k},\qquad \tau=\tau(k,\theta)=\frac{\sin k\theta}{k},$$
$$s=s(k,\theta)=\sigma+\mathrm{i}\tau=\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k},\qquad D(s)=s+\widehat{a}(s).$$

In the case $0\le\theta\le\varepsilon$, $c_0\theta\le\tau\le\theta$, $\sigma\le\tau$, (3.18) and (3.19) yield

$$|D(s)|\ge|\widehat{a}(\sigma+\mathrm{i}\tau)|-2\varepsilon\ge\frac{1}{2\sqrt{2}e}\int_0^{\frac{1}{\sqrt{2}\varepsilon}}a(t)\,\mathrm{d}t-2\varepsilon>\frac{1}{4\sqrt{2}e}\int_0^{\infty}a(t)\,\mathrm{d}t, \tag{3.23}$$

and

$$\int_0^{\varepsilon}|\widetilde{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta\le Ck^{-2}\int_0^{\varepsilon}\left(1+\int_0^{\frac{1}{\tau}}ta(t)\mathrm{d}t\right)\mathrm{d}\tau\le Ck^{-2}. \tag{3.24}$$

For $\frac{\varepsilon}{k}\le\theta\le\frac{\pi}{k}$, $\sigma(k,\theta)\ge\frac{1-\cos\varepsilon}{k}=\frac{c(\varepsilon)}{k}$, and from the proof of [1, Lemma 5], we know that $\mathrm{Re}\,\widehat{a}(\sigma+\mathrm{i}\tau)\ge 0$, so $|s+\widehat{a}(s)|\ge|\sigma+\mathrm{Re}\,\widehat{a}(\sigma+\mathrm{i}\tau)|\ge\sigma\ge\frac{c(\varepsilon)}{k}$. Furthermore, (1.11) gives us

$$|\widehat{a}(\sigma+\mathrm{i}\tau)|\le\frac{\delta}{|s|}+\frac{2}{|s|^2}\sum_{l=1}^{\infty}\frac{\delta_l}{l}\le Ck,\qquad |\widehat{a}_s'(\sigma+\mathrm{i}\tau)|\le\frac{1}{|s|^2}\left(2\delta+\frac{4}{|s|}\sum_{l=1}^{\infty}\frac{\delta_l}{l}\right)\le Ck^2,$$

from which it follows that

$$\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}|\widetilde{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta\le\frac{C}{k}\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}\frac{\frac{1}{k}+k+\frac{1}{k}k^2}{\left(\frac{c(\varepsilon)}{k}\right)^2}\mathrm{d}\theta\le Ck^{-1}. \tag{3.25}$$

For $\varepsilon\le\theta\le\frac{\varepsilon}{k}$, arguing similar to the proof of Lemma 3.1. Now we partition the interval into the following three sets

$$\left[\varepsilon,\frac{\varepsilon}{k}\right]=[\varepsilon,\theta_1]\cup[\theta_1,\overline{\theta_2}]\cup\left[\overline{\theta_2},\frac{\varepsilon}{k}\right],$$

where

$$\overline{\theta_2}=\frac{2}{\omega}\left(\delta+\frac{2}{\omega}\sum_{l=1}^{\infty}\frac{\delta_l}{l}\right)>2\omega.$$

On $[\varepsilon,\theta_1]$, $|D(s)|\ge C>0$, $|\widehat{a}(\sigma+\mathrm{i}\tau)|\le 4\int_0^{\frac{1}{c\varepsilon}}a(t)\,\mathrm{d}t$, $|\widehat{a}_s'(\sigma+\mathrm{i}\tau)|\le 40\int_0^{\frac{1}{\varepsilon c}}ta(t)\mathrm{d}t$, and

$$\int_{\varepsilon}^{\theta_1}|\widetilde{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta\le\frac{1}{k}\int_{\varepsilon}^{\theta_1}\frac{\frac{1}{k}+C+\frac{C}{k}}{C^2}\mathrm{d}\theta\le Ck^{-2}. \tag{3.26}$$

On $\left[\overline{\theta_2}, \frac{\varepsilon}{k}\right]$, we have that $|s| = \sqrt{\sigma^2+\tau^2} \geq \tau \geq c_0\theta > \omega$, where $\frac{1}{2} < c_0 \leq 1$, which, together with (1.11), implies

$$\begin{aligned}
|\widehat{a}(\sigma+\mathrm{i}\tau)| &\leq \frac{1}{\omega}\left(\delta + \frac{2}{\omega}\sum_{l=1}^{\infty}\frac{\delta_l}{l}\right),\\
|D(s)| &\geq |s| - |\widehat{a}(s)| \geq |s| - \frac{1}{\omega}\left(\delta + \frac{2}{\omega}\sum_{l=1}^{\infty}\frac{\delta_l}{l}\right)\\
&\geq \left(1-\frac{1}{2c_0}\right)|s| \geq \left(1-\frac{1}{2c_0}\right)\tau,
\end{aligned}$$

which yields

$$\int_{\overline{\theta_2}}^{\frac{\varepsilon}{k}} |\widetilde{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \leq \frac{C}{k}\int_{\overline{\theta_2}}^{\frac{\varepsilon}{k}} \frac{\frac{1}{k}+C+\frac{C}{k}}{\tau^2}\,\mathrm{d}\tau \leq Ck^{-2}. \tag{3.27}$$

Finally, for the interval $[\theta_1, \overline{\theta_2}]$, using $D(s) = (s-\mathrm{i}\omega)r(s)$, where $|r(s)| = |r(\sigma+\mathrm{i}\tau)| \geq C > 0$, and (3.4), we can get that

$$k^2\int_{\theta_1}^{\overline{\theta_2}} |\widetilde{u}_z'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta \leq Ck^{-2}. \tag{3.28}$$

Combining (3.21) and (3.24)–(3.28) we have established (3.20). This completes the proof of Lemma 3.2. □

Now we can prove Theorem 1 stated in introduction.

Proof of Theorem 1. Let $\widetilde{u_1}(z) = \widetilde{u}(z) - \overline{u}(z)$. We proceed as in the proof of Lemmas 3.1 and 3.2 and obtain

$$\sum_{n=1}^{\infty}|u_1^n| \leq 2k\int_{\theta_1}^{\overline{\theta_2}} |\widetilde{u}_{1,z}'(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta + Ck^{-1}. \tag{3.29}$$

Set $P(s) = \frac{1}{D(s)} - \frac{1}{\gamma(s-\mathrm{i}\omega)}$, and note that

$$D(\mathrm{i}\omega) = 0, \qquad D_s'(\mathrm{i}\omega) = \gamma. \tag{3.30}$$

Then with $s = \frac{1-z}{k}$, we have

$$\widetilde{u_1}(z) = \frac{z}{k}\left[P(s) - \frac{1}{\gamma(s+\mathrm{i}\omega)}\right] = \frac{z}{k}\widehat{u_1}(s). \tag{3.31}$$

Combining (3.29) and (3.31), and using Lemma 3.1 and the notations of Lemma 3.2 we find that

$$\sum_{n=1}^{\infty}|u_1^n| \leq 2k^{-1}\int_{\theta_1}^{\overline{\theta_2}} |P_s'(s)|\mathrm{d}\theta + Ck^{-1}. \tag{3.32}$$

Below we adapt the proof of [2, Lemma 3.1] to establish an estimate

$$\int_{\theta_1}^{\overline{\theta_2}} |P_s'(s)|\mathrm{d}\theta \leq C. \tag{3.33}$$

Following [2] we have

$$P_s'(s) = \frac{(s-\mathrm{i}\omega)^{-2}D^2(s) - \gamma D_s'(s)}{\gamma D^2(s)}, \quad s \neq \mathrm{i}\omega, s \in S = \{s|\mathrm{Re}\, s \geq 0, s \neq 0\}, \tag{3.34}$$

and

$$(s-\mathrm{i}\omega)^{-2}D^2(s) - \gamma D_s'(s) = D_s'(\mathrm{i}\omega)\left[2\frac{D(s)}{s-\mathrm{i}\omega} - (D_s'(s) + D_s'(\mathrm{i}\omega))\right] + O(s-\mathrm{i}\omega)^2, \quad (s \to \mathrm{i}\omega, s \in S). \tag{3.35}$$

By (1.11) and the hypothesis on $a(t)$,

$$D(s) = s + \widehat{a}(s) = (\mathrm{i}\omega)^{-2}b(s) + H(s),$$

where $H(s)$ is analytic in $\mathrm{Re}\, s > 0$ and $H^{(j)}(s), j = 0, 1, 2$ are continuous on $\{s|\mathrm{Re}\, s \geq 0, s \neq 0\}, H''(s) - H''(\mathrm{i}\omega) = O(s - \mathrm{i}\omega), (s \to \mathrm{i}\omega)$, and

$$b(s) = \sum_{l=1}^{\infty} \frac{\delta_l}{l}(\mathrm{e}^{-ls} - 1).$$

Then by Taylor's Theorem, we have

$$2\frac{H(s) - H(\mathrm{i}\omega)}{s - \mathrm{i}\omega} - (H'(s) + H'(\mathrm{i}\omega)) = O(s - \mathrm{i}\omega)^2, \quad (s \to \mathrm{i}\omega, s \in S). \tag{3.36}$$

On the other hand,

$$2\frac{b(s) - b(\mathrm{i}\omega)}{s - \mathrm{i}\omega} - (b'(s) + b'(\mathrm{i}\omega)) = \sum_{l=1}^{\infty} \mathrm{e}^{-\mathrm{i}l\omega}\left[\mathrm{e}^{-l(s-\mathrm{i}\omega)} + 1 + 2\frac{\mathrm{e}^{-l(s-\mathrm{i}\omega)} - 1}{l(s - \mathrm{i}\omega)}\right]\delta_l,$$

it follows that

$$\begin{aligned}
&\int_{\omega}^{\omega+\varepsilon} |(s - \mathrm{i}\omega)^{-2}| \left|2\frac{b(s) - b(\mathrm{i}\omega)}{s - \mathrm{i}\omega} - (b'(s) + b'(\mathrm{i}\omega))\right| \mathrm{d}\tau \\
&\leq C\int_{\omega}^{\omega+\varepsilon} |(s - \mathrm{i}\omega)^{-2}| \left[\sum_{l|s-\mathrm{i}\omega|\leq\varepsilon} l^2|s - \mathrm{i}\omega|^2\delta_l + \sum_{l|s-\mathrm{i}\omega|>\varepsilon} \delta_l\right] \mathrm{d}\tau \\
&= C\int_{\omega}^{\omega+\varepsilon} \left[\sum_{l|s-\mathrm{i}\omega|\leq\varepsilon} l^2\delta_l + \sum_{l|s-\mathrm{i}\omega|>\varepsilon} |(s - \mathrm{i}\omega)^{-2}|\delta_l\right] \mathrm{d}\tau \\
&\leq C\left(\sum_{l=1}^{\infty} \delta_l l^2 \int_0^{\frac{\varepsilon}{l}} \mathrm{d}\tau + \sum_{l=1}^{\infty} \delta_l \int_{l|s-\mathrm{i}\omega|>\varepsilon} \frac{\mathrm{d}\tau}{|s - \mathrm{i}\omega|^2}\right) \\
&\leq C\left(\sum_{l=1}^{\infty} \delta_l l + \sum_{l=1}^{\infty} \delta_l \left[\int_{\substack{l|s-\mathrm{i}\omega|>\varepsilon \\ l(\tau-\omega)>\frac{\varepsilon}{\sqrt{2}}}} + \int_{\substack{l|s-\mathrm{i}\omega|>\varepsilon \\ l(\tau-\omega)\leq\frac{\varepsilon}{\sqrt{2}}}} \frac{\mathrm{d}\tau}{|s - \mathrm{i}\omega|^2}\right]\right) \\
&\leq C\left(\int_0^{\infty} a(t)\mathrm{d}t + \sum_{l=1}^{\infty} \delta_l \int_{\tau-\omega>\frac{\varepsilon}{\sqrt{2}l}} \frac{\mathrm{d}\tau}{(\tau - \omega)^2} + \sum_{l=1}^{\infty} \delta_l \int_{\substack{\tau-\omega\leq\frac{\varepsilon}{\sqrt{2}l} \\ \sigma>\frac{\varepsilon}{\sqrt{2}l}}} \frac{\mathrm{d}\tau}{\sigma^2}\right) \\
&\leq C\int_0^{\infty} a(t)\, \mathrm{d}t < \infty.
\end{aligned} \tag{3.37}$$

Then (3.35)–(3.37) yield (3.33), and thus Theorem 1 is proved. □

4. Proof of Theorem 2

Similar to the argument of Section 3, from (1.12)–(1.15) we have with $\widetilde{U_1}(z) = \sum_{n=0}^{\infty} u_1^n z^n$,

$$\widetilde{U_1}(z) = \frac{z}{k}\widehat{u_1}\left(\frac{1 - z}{k}\right) + u_1^0. \tag{4.1}$$

Furthermore, by applying the z-transform formula (1.132) in [13], we can write that with $z = \mathrm{e}^{-kq}$, $\overline{U_1}(z) = \sum_{n=0}^{\infty} u_1(t_n)z^n$,

$$\overline{U_1}(z) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \widehat{u_1}(p)\frac{\mathrm{d}p}{1 - z\mathrm{e}^{kp}} + \frac{1}{2}u_1(0). \tag{4.2}$$

Differentiating (4.2) yields

$$\overline{U}'_{1,z}(z) = -\frac{1}{k^2}\sum_{l=-\infty}^{\infty} \mathrm{e}^{k\left(q+\frac{2l\pi}{k}\mathrm{i}\right)}\widehat{u_1}'\left(q + \frac{2l\pi}{k}i\right), \tag{4.3}$$

with $z = \mathrm{e}^{-kq}$, $\mathrm{Re}\, q \geq 0$. (For $\mathrm{Re}\, q = 0$, we have used a limit procedure, cf. [5,6].)

Differentiating (4.1) we get

$$\widetilde{U}'_{1,z}(z) = \frac{1}{k}\widehat{u_1}\left(\frac{1-z}{k}\right) - \frac{z}{k^2}\widehat{u_1}'\left(\frac{1-z}{k}\right). \tag{4.4}$$

Thus in a similar manner, we have

$$\sum_{n=1}^{\infty}|u_1(t_n) - u_1^n| \le 2k\int_0^{\frac{\pi}{k}} |\widetilde{E}'_z(\mathrm{e}^{-\mathrm{i}k\theta})|\mathrm{d}\theta, \tag{4.5}$$

where

$$\widetilde{E}'_z(z) = \overline{U}'_{1,z}(z) - \widetilde{U}'_{1,z}(z),$$

Combining (4.3) and (4.4) we can write $\widetilde{E}'_z$ as $\widetilde{E}'_z = \widetilde{E}'_{z,1} + \widetilde{E}'_{z,2} + \widetilde{E}'_{z,3} + \widetilde{E}'_{z,4}$ with $z = \mathrm{e}^{-kq}$, $q = \mathrm{i}\theta$,

$$\widetilde{E}'_{z,1}(q;k) = -\frac{1}{k^2}\sum_{l\ne 0}\mathrm{e}^{kq}\widehat{u_1}'\left(q + \frac{2l\pi}{k}\mathrm{i}\right),$$

$$\widetilde{E}'_{z,2}(q;k) = \frac{1}{k^2}\mathrm{e}^{-kq}\widehat{u_1}'(q) - \frac{1}{k^2}\mathrm{e}^{kq}\widehat{u_1}'(q),$$

$$\widetilde{E}'_{z,3}(q;k) = \frac{\mathrm{e}^{-kq}}{k^2}\widehat{u_1}'\left(\frac{1-\mathrm{e}^{-kq}}{k}\right) - \frac{\mathrm{e}^{-kq}}{k^2}\widehat{u_1}'(q),$$

$$\widetilde{E}'_{z,4}(q;k) = -\frac{1}{k}\widehat{u_1}\left(\frac{1-\mathrm{e}^{-kq}}{k}\right).$$

From (1.3), (1.13), (3.16) and (3.17) we find

$$\begin{aligned}\frac{1}{k}\sum_{l\ne 0}\int_0^{\frac{\pi}{k}}\left|\widehat{u_1}'\left(\mathrm{i}\theta + \frac{2l\pi}{k}\mathrm{i}\right)\right|\mathrm{d}\theta &= \frac{1}{k}\sum_{l\ne 0}\int_{\frac{2l\pi}{k}}^{\frac{2l\pi}{k}+\frac{\pi}{k}}|\widehat{u_1}'(\mathrm{i}\tau)|\mathrm{d}\tau \\ &\le \frac{2}{k}\int_{\frac{\pi}{k}}^{\infty}|\widehat{u_1}'(\mathrm{i}\tau)|\mathrm{d}\tau \le C.\end{aligned} \tag{4.6}$$

For $\widetilde{E}'_{z,2}$ we use Lemma 3.1 in [2] to get

$$\begin{aligned}\int_0^{\frac{\pi}{k}}|\theta\widehat{u_1}'(\mathrm{i}\theta)|\mathrm{d}\theta &= \int_0^{\overline{\theta_2}} + \int_{\overline{\theta_2}}^{\frac{\pi}{k}}|\theta\widehat{u_1}'(\mathrm{i}\theta)|\mathrm{d}\theta \\ &\le C\left(1 + \int_{\overline{\theta_2}}^{\frac{\pi}{k}}\frac{\mathrm{d}\theta}{\theta}\right) \le C|\log k|.\end{aligned} \tag{4.7}$$

According to the similar arguments to those in Lemmas 3.1 and 3.2, and applying the continuity of $\widehat{u_1}(q)$ in $\{\mathrm{Re}\, q \ge 0\}$, we have

$$\int_0^{\frac{\pi}{k}}\left|\widehat{u_1}\left(\frac{1-\mathrm{e}^{-\mathrm{i}k\theta}}{k}\right)\right|\mathrm{d}\theta \le C|\log k|. \tag{4.8}$$

It remains to establish an estimate on

$$k\int_0^{\frac{\pi}{k}}|\widetilde{E}'_{z,3}(\mathrm{i}\theta,k)|\mathrm{d}\theta \le C\int_0^{\frac{\varepsilon}{k}}|\theta^2\widehat{u_1}''(s_1)|\mathrm{d}\theta + k\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}}|\widetilde{E}'_{z,3}(\mathrm{i}\theta,k)|\mathrm{d}\theta \tag{4.9}$$

where we have used the mean theorem on $\left[0, \frac{\varepsilon}{k}\right]$, and with $s_1 = s_1(k,\theta) = \sigma_1(k,\theta) + \mathrm{i}\tau_1(k,\theta)$, and $0 \le \sigma_1 < \sigma(k,\theta)$, $\tau(k,\theta) \le \tau_1(k,\theta) < \theta$, and

$$\widehat{u_1}''(q) = \frac{2(1+\widehat{a}'(q))^2 - \widehat{a}''(q)(q+\widehat{a}(q))}{(q+\widehat{a}(q))^3} - \frac{2}{\gamma}\frac{2q^3 - 6q\omega^2}{(q^2+\omega^2)^3}.$$

Recall from [14, Lemma 6.1 (i)] that

$$|\widehat{a}''(\mathrm{i}\theta)| \le 600\int_0^{\infty}a(t)\mathrm{d}t\theta^{-2}, \quad (\theta > 0), \tag{4.10}$$

so that

$$|\widehat{a}''(\sigma+\mathrm{i}\theta)| \le 600\int_0^\infty a(t)\mathrm{d}t\theta^{-2}, \quad (\sigma\ge 0, \theta>0), \tag{4.11}$$

and combining (3.18), (3.19) and (4.11) we have

$$\widehat{u_1}''(\sigma+\mathrm{i}\theta) = O(\theta^{-3}) \qquad \theta\to\infty,\ \text{uniformly in the half plane } \sigma\ge 0. \tag{4.12}$$

Therefore, we again argue as in the Section 3 to obtain

$$\int_0^{\frac{\varepsilon}{k}} |\theta^2\widehat{u_1}''(s_1)|\mathrm{d}\theta \le C|\log k| + C\int_{\theta_1}^{\overline{\theta_2}} |P_s''(s_1)|\mathrm{d}\theta, \tag{4.13}$$

$$k\int_{\frac{\varepsilon}{k}}^{\frac{\pi}{k}} |\widetilde{E}'_{z,3}(\mathrm{i}\theta,k)|\mathrm{d}\theta \le C. \tag{4.14}$$

In Section 5 we shall prove that

$$\int_{\theta_1}^{\overline{\theta_2}} |P_s''(s_1)|\mathrm{d}\theta \le C. \tag{4.15}$$

From (4.5)–(4.9), and along with (4.13)–(4.15) we have now established that

$$\sum_{n=1}^{\infty} |u_1(t_n)-u_1^n| \le C|\log k|. \tag{4.16}$$

This completes the proof of Theorem 2. □

5. Proof of (4.15)

Differentiate the both sides of (3.34) to obtain, for $s\neq \mathrm{i}\omega$, $s\in S=\{s|\mathrm{Re}\, s\ge 0, s\neq 0\}$,

$$P''(s) = -\frac{1}{D'(\mathrm{i}\omega)D^3(s)}\left(D(s)D'(\mathrm{i}\omega)D''(s) + 2\frac{D^3(s)}{(s-\mathrm{i}\omega)^3} - 2D'(\mathrm{i}\omega)[D'(s)]^2\right). \tag{5.1}$$

Since

$$(s-\mathrm{i}\omega)^{-1}\left(D'(\mathrm{i}\omega) - \frac{D(s)}{s-\mathrm{i}\omega}\right) \to -\frac{1}{2}D''(\mathrm{i}\omega) \quad (s\to\mathrm{i}\omega, s\in S), \tag{5.2}$$

so that we have that, for $s\to\mathrm{i}\omega$, $s\in S$,

$$\frac{D^3(s)}{(s-\mathrm{i}\omega)^3} = 3\frac{D^2(s)}{(s-\mathrm{i}\omega)^2}D'(\mathrm{i}\omega) - 3\frac{D(s)}{s-\mathrm{i}\omega}[D'(\mathrm{i}\omega)]^2 + [D'(\mathrm{i}\omega)]^3 + O(s-\mathrm{i}\omega)^3, \tag{5.3}$$

and

$$\begin{aligned} &D(s)D'(\mathrm{i}\omega)D''(s) + 2\frac{D^3(s)}{(s-\mathrm{i}\omega)^3} - 2D'(\mathrm{i}\omega)[D'(s)]^2 \\ &\quad = D'(\mathrm{i}\omega)\left[6\frac{D(s)}{s-\mathrm{i}\omega}\left(\frac{D(s)}{s-\mathrm{i}\omega} - D'(\mathrm{i}\omega)\right) + D(s)D''(s) - 2(D'(s)+D'(\mathrm{i}\omega))(D'(s)-D'(\mathrm{i}\omega))\right] \\ &\qquad + O(s-\mathrm{i}\omega)^3, \quad (s\to\mathrm{i}\omega). \end{aligned} \tag{5.4}$$

By (2.2)–(2.6),

$$D(s) = (\mathrm{i}\omega)^{-2}b(s) + H(s), \tag{5.5}$$

where $H(s)$ is analytic in $\mathrm{Re}\, s>0$ and $H^{(j)}(s)$, $j=0,1,2,3$, are continuous on S, $H^{(3)}(s) - H^{(3)}(\mathrm{i}\omega) = O(s-\mathrm{i}\omega)$, $(s\to i\omega, s\in S)$, and

$$b(s) = \sum_{l=1}^{\infty}\frac{\delta_l}{l}(\mathrm{e}^{-ls}-1).$$

We put (5.5) into (5.4) to get

$$6\frac{D(s)}{s-\mathrm{i}\omega}\left(\frac{D(s)}{s-\mathrm{i}\omega}-D'(\mathrm{i}\omega)\right)+D(s)D''(s)-2(D'(s)+D'(\mathrm{i}\omega))(D'(s)-D'(\mathrm{i}\omega))=H(s,\mathrm{i}\omega)+HB(s,\mathrm{i}\omega)+B(s,\mathrm{i}\omega), \tag{5.6}$$

where

$$H(s,\mathrm{i}\omega)=6\frac{(H(s)-H(\mathrm{i}\omega))^2}{(s-\mathrm{i}\omega)^2}-6\frac{H(s)-H(\mathrm{i}\omega)}{s-\mathrm{i}\omega}H'(\mathrm{i}\omega)+(H(s)-H(\mathrm{i}\omega))H''(s)-2[(H'(s))^2-(H'(\mathrm{i}\omega))^2], \tag{5.7}$$

$$\begin{aligned}(\mathrm{i}\omega)^2HB(s,\mathrm{i}\omega) \;=\; & 12\frac{H(s)-H(\mathrm{i}\omega)}{s-\mathrm{i}\omega}\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-6\frac{H(s)-H(\mathrm{i}\omega)}{s-\mathrm{i}\omega}b'(\mathrm{i}\omega)\\ & -6H'(\mathrm{i}\omega)\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}+(H(s)-H(\mathrm{i}\omega))b''(s)\\ & +H''(s)(b(s)-b(\mathrm{i}\omega))-4\,[H'(s)b'(s)-H'(\mathrm{i}\omega)b'(\mathrm{i}\omega)],\end{aligned} \tag{5.8}$$

$$(\mathrm{i}\omega)^4B(s,\mathrm{i}\omega)=6\frac{(b(s)-b(\mathrm{i}\omega))^2}{(s-\mathrm{i}\omega)^2}-6\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}b'(\mathrm{i}\omega)+(b(s)-b(\mathrm{i}\omega))b''(s)-2[(b'(s))^2-(b'(\mathrm{i}\omega))^2]. \tag{5.9}$$

Then by Taylor's Theorem, there exist z_1, z_2, z_3 on the line from s to $\mathrm{i}\omega$ such that

$$H(s)=H(\mathrm{i}\omega)+H'(\mathrm{i}\omega)(s-\mathrm{i}\omega)+\frac{H''(\mathrm{i}\omega)}{2}(s-\mathrm{i}\omega)^2+\frac{H'''(z_1)}{6}(s-\mathrm{i}\omega)^3+O(s-\mathrm{i}\omega)^4, \tag{5.10}$$

$$H'(s)=H'(\mathrm{i}\omega)+H''(\mathrm{i}\omega)(s-\mathrm{i}\omega)+\frac{H'''(z_2)}{2}(s-\mathrm{i}\omega)^2+O(s-\mathrm{i}\omega)^3 \tag{5.11}$$

$$H''(s)=H''(\mathrm{i}\omega)+H'''(z_3)(s-\mathrm{i}\omega)+O(s-\mathrm{i}\omega)^2, \tag{5.12}$$

and

$$\begin{aligned}H(s,\mathrm{i}\omega) \;=\; & 6\frac{H(s)-H(\mathrm{i}\omega)}{s-\mathrm{i}\omega}\left[\frac{H''(\mathrm{i}\omega)}{2}(s-\mathrm{i}\omega)+\frac{H'''(z_1)}{6}(s-\mathrm{i}\omega)^2\right]+H''(s)\left[H'(\mathrm{i}\omega)(s-\mathrm{i}\omega)+\frac{H''(\mathrm{i}\omega)}{2}(s-\mathrm{i}\omega)^2\right]\\ & -2(H'(s)+H'(\mathrm{i}\omega))\left[H''(\mathrm{i}\omega)(s-\mathrm{i}\omega)+\frac{H'''(z_2)}{2}(s-\mathrm{i}\omega)^2\right]+O(s-\mathrm{i}\omega)^3\\ =\; & (s-\mathrm{i}\omega)H_1(s,\mathrm{i}\omega)+O(s-\mathrm{i}\omega)^3,\end{aligned} \tag{5.13}$$

with

$$\begin{aligned}H_1(s,\mathrm{i}\omega) \;=\; & 3\frac{H(s)-H(\mathrm{i}\omega)}{s-\mathrm{i}\omega}H''(\mathrm{i}\omega)+\frac{H(s)-H(\mathrm{i}\omega)}{s-\mathrm{i}\omega}H'''(z_1)(s-\mathrm{i}\omega)+H''(s)H'(\mathrm{i}\omega)+\frac{H''(\mathrm{i}\omega)}{2}H''(s)(s-\mathrm{i}\omega)\\ & -2(H'(s)+H'(\mathrm{i}\omega))H''(\mathrm{i}\omega)-2(H'(s)+H'(\mathrm{i}\omega))\frac{H'''(z_2)}{2}(s-\mathrm{i}\omega)\\ =\; & (s-\mathrm{i}\omega)H_2(s,\mathrm{i}\omega)+O(s-\mathrm{i}\omega)^2,\end{aligned} \tag{5.14}$$

where

$$\begin{aligned}H_2(s,\mathrm{i}\omega) \;=\; & \frac{1}{2}H''(\mathrm{i}\omega)(H''(s)-H''(\mathrm{i}\omega))+H'(\mathrm{i}\omega)(H'''(z_3)-H'''(z_2))+\frac{H(s)-H(\mathrm{i}\omega)}{s-\mathrm{i}\omega}H'''(z_1)-H'(s)H'''(z_2)\\ =\; & \frac{1}{2}H''(\mathrm{i}\omega)(H''(s)-H''(\mathrm{i}\omega))+H'(\mathrm{i}\omega)(H'''(z_3)-H'''(z_2))\\ & +H'(\mathrm{i}\omega)(H'''(z_1)-H'''(z_2))+O(s-\mathrm{i}\omega)=O(s-\mathrm{i}\omega).\end{aligned} \tag{5.15}$$

Thus, combining (5.13)–(5.15) we obtain that

$$H(s,\mathrm{i}\omega)=O(s-\mathrm{i}\omega)^3,\quad (s\to\mathrm{i}\omega). \tag{5.16}$$

Similarly,

$$(\mathrm{i}\omega)^2HB(s,\mathrm{i}\omega)=H'(\mathrm{i}\omega)HB_1(s,\mathrm{i}\omega)+H''(\mathrm{i}\omega)HB_2(s,\mathrm{i}\omega)+HB_3(s,\mathrm{i}\omega)+O(s-\mathrm{i}\omega)^3, \tag{5.17}$$

where

$$HB_1(s,\mathrm{i}\omega)=6\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right]+(s-\mathrm{i}\omega)b''(s)-4(b'(s)-b'(\mathrm{i}\omega)), \tag{5.18}$$

$$\begin{aligned}HB_2(s,\mathrm{i}\omega)\;&=\;3(s-\mathrm{i}\omega)\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right]+3(s-\mathrm{i}\omega)\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}\right]\\&\quad+\frac{(s-\mathrm{i}\omega)^2}{2}b''(s)+(b(s)-b(\mathrm{i}\omega))-4(s-\mathrm{i}\omega)b'(s),\end{aligned} \tag{5.19}$$

$$\begin{aligned}HB_3(s,\mathrm{i}\omega)\;&=\;(s-\mathrm{i}\omega)^2\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right]H'''(z_1)+(s-\mathrm{i}\omega)^2\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}\right]H'''(z_1)\\&\quad+(s-\mathrm{i}\omega)(b(s)-b(\mathrm{i}\omega))H'''(z_3)-4(s-\mathrm{i}\omega)^2b'(s)\frac{H'''(z_2)}{2}.\end{aligned} \tag{5.20}$$

In order to obtain estimates on HB_2 and HB_3 we further use Taylor's Theorem to get that there exist z_4, z_5, z_6 on the line from s to $i\omega$ such that (5.10)–(5.12) hold, with b instead of H, and z_4, z_5, z_6 instead of z_1, z_2, z_3, respectively. Using these expressions we find that

$$HB_2(s,\mathrm{i}\omega)=(s-\mathrm{i}\omega)HB_{2,1}(s,\mathrm{i}\omega)+O(s-\mathrm{i}\omega)^3, \tag{5.21}$$

with

$$\begin{aligned}HB_{2,1}(s,\mathrm{i}\omega)\;&=\;3\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right]+3\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}+\frac{1}{2}(s-\mathrm{i}\omega)\,b''(s)\\&\quad+b'(\mathrm{i}\omega)+\frac{b''(\mathrm{i}\omega)}{2}(s-\mathrm{i}\omega)-4b'(s)=\frac{s-\mathrm{i}\omega}{2}(b''(s)-b''(\mathrm{i}\omega))+O(s-\mathrm{i}\omega)^2=O(s-\mathrm{i}\omega)^2,\end{aligned} \tag{5.22}$$

and adding that H''' is Lipschitz continuous on S to (5.20) yields

$$\begin{aligned}HB_3(s,\mathrm{i}\omega)\;&=\;H'''(\mathrm{i}\omega)(s-\mathrm{i}\omega)\Bigg[(s-\mathrm{i}\omega)\left\{\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right\}\\&\quad+(s-\mathrm{i}\omega)\left\{\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}\right\}+b(s)-b(\mathrm{i}\omega)-2(s-\mathrm{i}\omega)b'(s)\Bigg]\\&\quad+O(s-\mathrm{i}\omega)^3=H'''(\mathrm{i}\omega)(s-\mathrm{i}\omega)HB_{3,1}(s,\mathrm{i}\omega)+O(s-\mathrm{i}\omega)^3,\end{aligned} \tag{5.23}$$

where

$$HB_{3,1}(s,\mathrm{i}\omega)=(s-\mathrm{i}\omega)\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)+\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}+b'(\mathrm{i}\omega)-2b'(s)\right]+O(s-\mathrm{i}\omega)^2=O(s-\mathrm{i}\omega)^2. \tag{5.24}$$

Since $b'(s)=-\sum_{l=1}^{\infty}\delta_l\,\mathrm{e}^{-ls}$, $b''(s)=\sum_{l=1}^{\infty}l\delta_l\,\mathrm{e}^{-ls}$,

$$|HB_1(s,\mathrm{i}\omega)|\le\sum_{l=1}^{\infty}\delta_l|\mathrm{e}^{-\mathrm{i}l\omega}|\left|6\frac{\mathrm{e}^{-l(s-\mathrm{i}\omega)}-1}{l(s-\mathrm{i}\omega)}+2+(s-\mathrm{i}\omega)l\,\mathrm{e}^{-l(s-\mathrm{i}\omega)}+4\mathrm{e}^{-l(s-\mathrm{i}\omega)}\right|.$$

The function $J(s,\mathrm{i}\omega)$ inside the absolute value signs in this integral can be written $J(s,\mathrm{i}\omega)=K(l(s-\mathrm{i}\omega))$, where $K(x)$ is a C^3 function with $K(0)=K'(0)=K''(0)=0$, $K'''(0)=\frac{1}{2}$, and $|K(x)|\le C|x|$ for all $\mathrm{Re}\,x\ge 0$. We choose $\mu>0$ such that $|K(x)|\le|x|^3$ if $|x|\le\mu$. Then by Fubini's Theorem and (2.5) we have that

$$\begin{aligned}\int_{\theta_1}^{\overline{\theta_2}}|(s_1-\mathrm{i}\omega)^{-3}|HB_1(s_1,\mathrm{i}\omega)|\mathrm{d}\theta\;&\le\;C\int_{\theta_1}^{\overline{\theta_2}}|(s_1-\mathrm{i}\omega)^{-3}|\left[\sum_{l|s_1-\mathrm{i}\omega|\le\mu}l^3|s_1-\mathrm{i}\omega|^3\delta_l+\sum_{l|s_1-\mathrm{i}\omega|>\mu}l|s_1-\mathrm{i}\omega|\delta_l\right]\mathrm{d}\theta\\&\le\;C\int_{\omega}^{\omega+\varepsilon}\left[\sum_{l|s-\mathrm{i}\omega|\le c}l^3\delta_l+\sum_{l|s-\mathrm{i}\omega|>c}|(s-\mathrm{i}\omega)^{-2}|l\,\delta_l\right]\mathrm{d}\tau\\&\le\;C\left(\sum_{l=1}^{\infty}\delta_l l^3\int_0^{\frac{c}{l}}\mathrm{d}\tau+\sum_{l=1}^{\infty}l\delta_l\int_{l|s-\mathrm{i}\omega|>c}\frac{\mathrm{d}\tau}{|s-\mathrm{i}\omega|^2}\right)\\&\le\;C\left(\sum_{l=1}^{\infty}\delta_l l^2+\sum_{l=1}^{\infty}l\delta_l\left[\int_{\substack{l|s-\mathrm{i}\omega|>c\\ l(\tau-\omega)>\frac{c}{\sqrt{2}}}}+\int_{\substack{l|s-\mathrm{i}\omega|>c\\ l(\tau-\omega)\le\frac{c}{\sqrt{2}}}}\frac{\mathrm{d}\tau}{|s-\mathrm{i}\omega|^2}\right]\right)\end{aligned}$$

$$\leq C\left(\int_0^\infty ta(t)\mathrm{d}t+\sum_{l=1}^{\infty} l\delta_l\int_{\tau-\omega>\frac{c}{l}}\frac{\mathrm{d}\tau}{(\tau-\omega)^2}+\sum_{l=1}^{\infty} l\delta_l\int_{\substack{\tau-\omega\leq\frac{c}{l}\\ \sigma>\frac{c}{l}}}\frac{\mathrm{d}\tau}{\sigma^2}\right)$$
$$\leq C\int_0^\infty ta(t)\,\mathrm{d}t<\infty. \tag{5.25}$$

Now we need establish an estimate on $B(s,\mathrm{i}\omega)$. In a similar manner, we see that

$$\begin{aligned}(\mathrm{i}\omega)^4B(s,\mathrm{i}\omega) &= 6\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right]\left[b'(\mathrm{i}\omega)+\frac{b''(\mathrm{i}\omega)}{2}(s-\mathrm{i}\omega)\right.\\ &\quad\left.+\frac{b'''(z_4)}{6}(s-\mathrm{i}\omega)^2\right]+b''(s)\left[b'(\mathrm{i}\omega)(s-\mathrm{i}\omega)+\frac{b''(\mathrm{i}\omega)}{2}(s-\mathrm{i}\omega)^2\right]\\ &\quad-2(b'(s)-b'(\mathrm{i}\omega))\left[2b'(\mathrm{i}\omega)+b''(\mathrm{i}\omega)(s-\mathrm{i}\omega)+\frac{b'''(z_5)}{2}(s-\mathrm{i}\omega)^2\right]+O(s-\mathrm{i}\omega)^3\\ &= b'(\mathrm{i}\omega)HB_1(s,\mathrm{i}\omega)+b''(\mathrm{i}\omega)B_1(s,\mathrm{i}\omega)+B_2(s,\mathrm{i}\omega)+O(s-\mathrm{i}\omega)^3,\end{aligned} \tag{5.26}$$

where

$$\begin{aligned}B_1(s,\mathrm{i}\omega) &= (s-\mathrm{i}\omega)\left[3\left(\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right)+\frac{1}{2}(s-\mathrm{i}\omega)b''(s)-2(b'(s)-b'(\mathrm{i}\omega))\right]\\ &= (s-\mathrm{i}\omega)\left[\frac{3}{2}b''(\mathrm{i}\omega)(s-\mathrm{i}\omega)+\frac{(s-\mathrm{i}\omega)}{2}b''(\mathrm{i}\omega)-2(s-\mathrm{i}\omega)b''(\mathrm{i}\omega)\right]+O(s-\mathrm{i}\omega)^3\\ &= O(s-\mathrm{i}\omega)^3,\end{aligned} \tag{5.27}$$

and

$$B_2(s,\mathrm{i}\omega)=b'''(z_4)(s-\mathrm{i}\omega)^2\left[\frac{b(s)-b(\mathrm{i}\omega)}{s-\mathrm{i}\omega}-b'(\mathrm{i}\omega)\right]-b'''(z_5)(s-\mathrm{i}\omega)^2[b'(s)-b'(\mathrm{i}\omega)]=O(s-\mathrm{i}\omega)^3. \tag{5.28}$$

From (5.1), (5.4), (5.6) and (5.16)–(5.28) we have proved (4.15). □

6. Further remark on the results

(i) We guess that a finer analysis reveals that the bound in Theorem 2 can be remove logarithm factor $|\log k|$.

(ii) The analysis carried out in this paper for the problem (1.1)-(1.4) is applicable, with very minor modification, to more general equation

$$u'(t)+\int_0^t[d+a(t-s)]u(s)\mathrm{d}s=0.\quad t>0,u(0)=1, \tag{6.1}$$

where $d\geq 0$ is a constant and the kernel a has more general form

$$a(t)=\sum_{l=1}^{\infty}\delta_l\left(1-\frac{\min\{t,lt_0\}}{lt_0}\right) \tag{6.2}$$

with $t_0>0$, (1.3) and

$$\omega\equiv\sqrt{\delta+d}=2\pi j/t_0,\quad \text{for some integers } j. \tag{6.3}$$

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References

[1] K.B. Hannsgen, Indirect abelian theorems and a linear Volterra equation, Trans. Amer. Math. Soc. 142 (1969) 539–555.
[2] K.B. Hannsgen, An L^1 remainder theorem for an integrodifferential equation with asymptotically periodic solution, Proc. Amer. Math. Soc. 73 (1979) 331–337.
[3] Ch. Lubich, Convolution quadrature and discretized operational calculus I, Numer. Math. 52 (1988) 129–145.
[4] R.W. Carr, K.B. Hannsgen, A nonhomogeneous integrodifferential equation in Hilbert space, SIAM J. Math. Anal. 10 (1979) 961–984.
[5] Da Xu, The global behavior of time discretization for an abstract Volterra equation in Hilbert space, Calcolo 34 (1997) 71–104.
[6] Da Xu, The long time global behavior of time discretization for fractional order Volterra equations, Calcolo 35 (1998) 93–116.
[7] Da Xu, Uniform l^1 behavior for time discretization of a Volterra equation with completely monotonic kernel: I. Stability, IMA J. Numer. Anal. 22 (2002) 133–151.
[8] Da Xu, Uniform L^1 error bounds for the semidiscrete solution of a Volterra equation with completely monotonic convolution kernel, Comput. Math. Appl. 43 (2002) 1303–1318.
[9] Da Xu, Stability of the difference type methods for linear Volterra equations in Hilbert spaces, Numer. Math. 109 (2008) 571–595.
[10] Da Xu, Uniform l^1 behavior for time discretization of a Volterra equation with completely monotonic kernel: II. Convergence, SIAM J. Numer. Anal. 46 (2008) 231–259.
[11] P.L. Duren, The Theory of H^p Spaces, Academic Press, New York, 1970.
[12] K.B. Hannsgen, Uniform boundedness in a class of Volterra equations, SIAM J. Math. Anal. 6 (1975) 689–697.
[13] E.I. Jury, Theory and Application of the z-Transform Method, John Wiley, Sons. Inc., New York, 1964.
[14] K.B. Hannsgen, Uniform L^1 behavior for an integro-differential equation with parameter, SIAM J. Math. Anal. 8 (1977) 626–639.